ON A NEW PROOF OF THE INFINITUDE OF THE PRIMES

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ABSTRACT. — A certain (multiplicative) number-theoretic function, Ψ, associated in a natural way with a new model of the Fundamental Theorem of Arithmetic is sufficiently well endowed with *invariance* characteristics so as to provide the basis for another analytic proof of Euclid's Theorem.

The purpose of this note is to relate a theorem which dates from antiquity (Euclid of Alexandria, *c.* 300 B.C.) with the author's recent reformulation of the *Unique Factorization Theorem* of elementary number theory (Mullin, 1963a, 1963b). The proof is nonelementary in that it uses ideas from analysis; specifically it uses the existence of the limiting density of fixed-points for a certain (multiplicative) number-theoretic function, Ψ, to be defined later. Furthermore the proof is distinct from the analytic proofs given by Vinogradov (Vinogradov, 1954, p. 151-152). However, it is not *necessary* to use the Ψ-function to prove Euclid's theorem any more than it is necessary to use Riemann's zeta-function. The claim to novelty for this note is that the number-theoretic function, Ψ, rich in properties and occurring quite naturally in the context of a *new* model of the Fundamental Theorem of Arithmetic provides such natural structure (specifically that associated with the useful algebraic notion of *invariance*) as to permit a proof of Euclid's Theorem.

As a point of entry, consider Gauss' model of the *Fundamental Theorem* (Gauss, 1801), *viz.*, every natural number n has a *unique* representation in the form $n = p_1\ \alpha_1 \ldots p_m\ \alpha_m$, where the p_i are *distinct* primes. Suppose, for natural number n, one applies Gauss' model to its own natural number exponents and to their exponents etc. (i.e., use induction) until the process terminates by the Well-ordering Principle. Call the final *unique* configuration of primes alone a *mosaic*; for example, the mosaic of 400 is $2^{2^2} \cdot 5^2$. Now define the (multiplicative) number-theoretic function Ψ as follows: Ψ(n) is the simple *product* of the primes alone in the mosaic of n. E.g., Ψ(400) = 80.

Lemma. Let n be a natural number > 1. Then Ψ(n) = n if, and alone in the mosaic of n. E.g., Ψ(400) = 80.

Proof. First note that Ψ(n) ≤ n for every natural number n. Clearly any square-free number is invariant under Ψ. Twice an *even* square-free is invariant under Ψ since Ψ(4) = 4 and every (odd) square-free number is invariant. No other natural number is invariant since one is changing an exponent into a multiplicative factor which strictly decreases the number.

Theorem. There exist infinitely many primes.

Proof. Suppose there are only finitely many primes. Then, by the *Fundamental Theorem* (whose proof is independent of the infinitude of the primes) there would be only finitely many square-free numbers. Hence, by the Lemma, only finitely many natural numbers would be invariant under Ψ. Let α be the distribution function of the fixed-points for Ψ. Hence, with $n \to \infty$, $\lim \alpha(n)/n = 0$. But, with $n \to \infty$, $\lim \alpha(n)/n = \lim (Q(n)/n + (1/2)(1/3)Q(n)/n) = 6/\pi^2 + 1/\pi^2 = 7/\pi^2$, where Q is the distribution of the square-free numbers. The latter result follows from Euler's Identity whose proof is independent of the infinitude of the primes. Hence, by *reductio ad absurdum*, there are infinitely many primes.

Literature Cited

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