

A CONCEPT CONCERNING A SET WITH A BINARY COMPOSITION LAW

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INTRODUCTION

This paper deals with a development of a side-problem concerned with the author's physico-logical researches in biological computation (Mullin, 1958, 1959, 1960). In particular, the paper is concerned with a unifying concept associated with sets that possess a closed binary composition law.

Basically we will concern ourselves with anticlosed subsets of closed systems. Some pertinent theorems are developed for general systems and systems with the structure of a group.

GENERAL CASE

Definition 1.1: Let $(B, *)$ denote a non-empty set B together with a closed binary composition law "*" defined on B . Call $(B, *)$ an algebraic system. By a *mutant* of $(B, *)$ is meant a subset M of B which satisfies the condition

$$M * M \subseteq \bar{M},$$

where $M * M = \{a * b : a \in M, b \in M\}$ and \bar{M} is the set of all of the elements of B not in M . Clearly M must be a proper subset of B . If all of the elements of B are idempotent with respect to "*" then the empty set is to be considered the only mutant of $(B, *)$. With this convention every algebraic system has a mutant.

Theorem 1.1: Every subset of a mutant M of $(B, *)$ is a mutant of $(B, *)$.

Proof: $A * A \subseteq M * M \subseteq \bar{M} \subseteq \bar{A}$, for every $A \subseteq M$.

Theorem 1.2: Let $\{S_i : i \in I\}$ be a non-empty class of mutants of $(B, *)$. Then

$$(i) \bigcap_{i \in I} S_i \text{ is a mutant of } (B, *)$$

and

$$(ii) \left(\bigcap_{i \in I} S_i \right) * \left(\bigcap_{i \in I} S_i \right) \subseteq \bigcap_{j \in I} \bar{S}_j.$$

Theorem 1.3: Let φ be a homomorphism from $(A, *)$ into (B, \circ) . Let M be a mutant of $(A, *)$. If $\varphi(\bar{M}) \subseteq \overline{\varphi(M)}$ then $\varphi(M)$ is a mutant of (B, \circ) .

Proof: $\varphi(M) \circ \varphi(M) = \varphi(M * M) \subseteq \varphi(\bar{M}) \subseteq \overline{\varphi(M)}$.

Theorem 1.4: Let φ be a homomorphism from $(A, *)$ onto (B, \circ) . Let M be a mutant of (B, \circ) . Let C be the inverse image of M under the homomorphism. Then C is a mutant of $(A, *)$.

Proof: for each $d_i \in M$ there exists a $c_j \in C$ such that $\varphi(c_j) = d_i$. But $\varphi(c_k * c_l) = \varphi(c_k) \circ \varphi(c_l) \in \bar{M}$. Then $c_k * c_l \notin C$, for otherwise, $\varphi(c_k * c_l) \in M$, which is a contradiction.

Theorem 1.5: Let M be a mutant of (B, \circ) . Then there is no homomorphism from a closed subset of S of \bar{M} into M .

Proof: Assume that there exists such a homomorphism φ . Then for $s_1 \in S$ and $s_2 \in S$, $\varphi(s_1 \circ s_2) = \varphi(s_1) \circ \varphi(s_2) \in \bar{M}$. But since $s_1 \circ s_2 \in S$, $\varphi(s_1 \circ s_2) \in M$.

Theorem 1.6: Let φ be an isomorphism from $(A, *)$ onto (B, \circ) . Then each mutant in either system induces a mutant in the other system.

Definition 1.2: A mutant M of (B, \circ) is said to be a *maximal mutant* of (B, \circ) provided there is no mutant of (B, \circ) which properly contains M . If all of the elements of B are idempotent with respect to "o" then the empty set is to be considered the only maximal mutant of (B, \circ) . With this convention and using Zorn's lemma it is clear that every algebraic system has a maximal mutant.

Theorem 1.7: Every mutant of (B, o) is contained in some maximal mutant of (B, o) .

Proof: Apply the Kuratowski Lemma, which states that each chain of a partially ordered set (set inclusion in our case) is contained in a maximal chain.

Theorem 1.8: If N is a proper subset of a mutant M of (B, o) then N is not a maximal mutant of (B, o) .

Theorem 1.9: Let φ be an isomorphism from $(A, *)$ onto (B, o) . Then each maximal mutant in either system induces a maximal mutant in the other system.

GROUPS

In this case we assume $(A, *)$ has the algebraic structure of a group. As an example put $(G, +)$ equal to the additive group of integers. Then the set of all odd integers is a maximal mutant of $(G, +)$ and the set of all even integers is a subgroup of $(G, +)$.

Theorem 2.1: Let (G, o) be a group. Then the only non-mutant (left or right) coset of a subgroup (H, o) is H .

Theorem 2.2: Any group isomorphic to the additive group of integers (A) has no finite maximal mutants, (B) has infinite maximal mutants which

- (i) have no elements in common
- (ii) have a finite number of elements in common
- (iii) have an infinite number of elements in common

Theorem 2.3: Any group isomorphic to the additive group $(G, +)$ of integers has a collection C of mutants such that the cardinality of C is that of the continuum.

SUMMARY

This paper defines a new concept concerning a set with a binary composition law. The concept is a relative anticlosure property. A number of theorems are developed, which show how the concept leads to new mathematics and how the concept is related to other mathematics.

LITERATURE CITED

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