# A RESIDUE TEST FOR BOOLEAN FUNCTIONS

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## INTRODUCTION

A residue test is a convenient means for determining the essential literals (Quine, 1952) in the transmission function describing the electrical properties of a two-terminal combinational switching circuit (Shannon, 1949), and thus indicates which literals are absolutely necessary in a realization of the switching circuit.

The residue test is based upon a pair of identities of the Boolean algebra related by duality (Birkhoff and MacLane, 1953). These identities are 1 and 1′ below.

The validity of these identities follows immediately by using the method of perfect induction with respect to the variable $x_k$. Note: "+" (sum) is used instead of "$U$" (cup), "." (product) instead of "$\Omega$" (cap) and a prime (complement) instead of "$\sim$" (negative).

For the purpose of brevity definitions (2, below) are made.

Hence $T_{lk}$ merely denotes the original $T$ with the particular variable $x_k$ in the $l$ state and similarly $T_{ok}$ denotes the original $T$ with the particular variable $x_k$ in the $O$ state.

With the definitions given by the identities of (2), the identities given by (1) and 1′) become, respectively, 3 and 3′, below.

At this point, we define $T_{lk}$ and $T_{ok}$ to be the residues of $T$ with respect to $x_k$ and $x_k'$, respectively.

In what follows, if no composition law is given, the composition "." (product) is implied. Example 1.

$$(1)\qquad T(x_n,\cdots,x_{k-1},x_k,x_{k-1},\cdots,x_0) \equiv x_k\cdot T(x_n,\cdots,x_{k+1},1,x_{k-1},\cdots,x_0) + \\ + x_k'\cdot T(x_n,\cdots,x_{k-1},0,x_{k-1},\cdots,x_0)$$

$$(1')\qquad T(x_n,\cdots,x_{k-1},x_k,x_{k-1},\cdots,x_0) \equiv [x_k + T(x_n,\cdots,x_{k-1},0,x_{k-1},\cdots,x_0]\cdot \\ \cdot[x_k' + T(x_n,\cdots,x_{k-1},1,x_{k-1},\cdots,x_0]$$

$$(2)\qquad \begin{aligned} T &\equiv T(_n,\cdots,x_{k-1},x_k,x_{k-1},\cdots,x_0) \\ T_{1k} &\equiv T(x_n,\cdots,x_{k-1},1,x_{k-1},\cdots,x_0) \qquad k=0,1,\cdots,n \\ T_{0k} &\equiv T(x_n,\cdots,x_{k+1},0,x_{k+1},\cdots,x_0) \end{aligned}$$

$$(3)\qquad T \equiv x_k\cdot T_{1k} + x_k'\cdot T_{0k}$$

$$(3')\qquad T \equiv [x_k + T_{0k}]\cdot[x_k' + T_{1k}] \qquad k=0,1,\cdots,n$$

Example 1:

*Let* $T(x_3,x_2,x_1,x_0) \equiv x_3'\,x_2'\,x_1\,x_0' + x_3\,x_2\,x_1\,x_0 + x_3'\,x_2\,x_1\,x_0' + \\ + x_3\,x_2\,x_1'\,x_0$

Inspect $x_1^*$ for its necessary appearance in $T$.

$$T \equiv x_1\, T_{11} + x_1'\, T_{01}$$
$$T \equiv x_1\, [x_3'\, x_2'\, x_0' + x_3\, x_2\, x_0 + x_3'\, x_2\, x_0'] + x_1'\, [x_3\, x_2\, x_0]$$
$$T \equiv x_1[x_3'\, x_0' + x_3\, x_2\, x_0] + x_1'\, [x_3\, x_2\, x_0]$$
$$T \equiv x_3\, x_2\, x_0 + x_3'\, x_1\, x_0'$$

This indicates that the literal $x'_1$ is unnecessary in a representation of $T$, since it combined with $x_1$ according to the identity

$$x_j\, x_k + x_j\, x_k' \equiv x_j$$

Given any transmission function $T$, the results of the residue test obtained by means of applying the identities given by (3) can be summarized. The results of the residue test obtained by means of applying the identities given by (3′) is equivalent and need not be carried out here.

Case 1:
$T \equiv x_k\, T_{1k} + x_k'\, T_{0k} \equiv T_{0k} \equiv T_{1k}$

Here neither $x_k$ nor $x_k'$ is necessary for a realization of $T$.

Case 2:
(a) $T_{1k} \supset T_{0k}, T_{1k}$ is said to contain $T_{0k}$; that is, $T_{1k} \not\equiv T_{0k}$ but if $T_{0k} \equiv 1$ then $T_{1k} = 1$. Under the condition, $T_{0k}\, T_{1k} \equiv T_{0k}$ and $T_{0k} + T_{1k} \equiv T_{1k}$. Therefore, $T \equiv x_k\, T_{1k} + x_k'\, T_{0k} \equiv x_k\, (T_{0k} + T_{1k}) + x_k'\, T_{0k}$; $T \equiv x_k\, T_{1k} + T_{0k}$
Hence $x_k'$ is not necessary for a realization of $T$.

(b) $T_{0k} \supset T_{1k}, T_{0k}$ is said to contain $T_{1k}$; that is, $T_{1k} \not\equiv T_{0k}$ but if $T_{1k} = 1$ then $T_{0k} = 1$. By reasoning similar to case (2a), the variable $x_k$ is not necessary in a realization of $T$.

Case 3:
Neither of the above cases. For this case no reduction in the number of literals is possible and hence both $x_k$ and $x_k'$ are necessary in a realization of $T$.

In general, it can be stated that the literal which is the coefficient of a residue that is contained in the other residue is not necessary in a realization of $T$.

A more fundamental approach to the problem of residue evaluation, and one that eventually leads to a complete residue test by means of a few simple rules, is based upon a matrix of combinations for $T$.

Given a transmission function $T$, we can construct a matrix, from the $(n+1)$ variables of $T$, with $(n+1)$ columns and $2^{(n+1)}$ rows such that one variable heads each column and each row is one of the $2^{(n+1)}$ possible states of the variables, taken collectively. Pre-augment the matrix by a $d$-column that has for elements lying in a given row of the original matrix, the decimal equivalent of the binary number representation of that row. Post-augment the latter matrix with a $T$-column that has for elements in a given row of the original matrix the state of $T$ corresponding to the state of the variables agreeing with that row of the matrix.

Using the above procedure, example 1 becomes the following matrix:

| $d$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $T$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 1 | 1 | 0 |
| 4 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 1 | 0 | 1 | 0 |
| 6 | 0 | 1 | 1 | 0 | 1 |
| 7 | 0 | 1 | 1 | 1 | 0 |
| 8 | 1 | 0 | 0 | 0 | 0 |
| 9 | 1 | 0 | 0 | 1 | 0 |
| 10 | 1 | 0 | 1 | 0 | 0 |
| 11 | 1 | 0 | 1 | 1 | 0 |
| 12 | 1 | 1 | 0 | 0 | 0 |
| 13 | 1 | 1 | 0 | 1 | 1 |
| 14 | 1 | 1 | 1 | 0 | 0 |
| 15 | 1 | 1 | 1 | 1 | 1 |

In general, the matrix of combinations will be a $2^{n+1}$x $(n+3)$ matrix.

If $p$ of the elements in the $T$-column are in the 1 state the $2^{n+1}—p$ of the elements in the $T$-column are in the $0$ state. Disregard the rows of the matrix of combinations that have a *0* in the $T$-column. Nothing is lost by doing this since we know the states of $x_n, \cdots, x_o$ for which $T = 1$ and $T$ must be *0* for all other states of $x_n, \cdots, x_o$. At this point a $p$x $(n+3)$ matrix remains. In example 1 this matrix becomes:

| $d$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $T$ |
|---|---|---|---|---|---|
| 2 | 0 | 0 | 1 | 0 | 1 |
| 6 | 0 | 1 | 1 | 0 | 1 |
| 13 | 1 | 1 | 0 | 1 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 |

If $q$ of the elements of the $p$x $(n+3)$ matrix have a *1* in the $x_k$ column, where $x_k$ is the variable being tested for necessary appearance in a representation of $T$, then $(p\text{-}q)$ of the elements in the $x_k$ column are *0*. Shift the rows of the $p$x $(n+3)$ matrix so as to partition it into $q$ rows with all *1*'s in the $x_k$ column and the remaining $(p\text{-}q)$ rows with all *0*'s in the $x_k$ column.

In example 1 this matrix is (upon inspecting the $x_1$ column):

| $d$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $T$ |
|---|---|---|---|---|---|
| 2 | 0 | 0 | 1 | 0 | 1 |
| 6 | 0 | 1 | 1 | 0 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 0 | 1 | 1 |

By disregarding the $x_1$ column of the partitioned matrix, since the necessary appearance of $x_1$* is being tested, it is seen that

$$T_{11} \equiv x_3' x_2' x_0' + x_3' x_2 x_0' + x_3 x_2 x_0$$

and

$$T_{01} \equiv x_3 x_2 x_0$$

where $T_{11}$ and $T_{01}$ have as many terms as there are rows in the partitioned matrix with *1*'s and *0*'s respectively, in the $x_1$ column. Each term of $T_{11}$ and $T_{01}$ corresponds to a row of the partitioned matrix and is composed of the product of all the variables except $x_1$, with the variable unprimed if a *1* is in the column corresponding to the particular variable and with the variable primed if a *0* is in the column corresponding to the particular variable. By using the results of case 2 of the residue test given previously, it is seen that no $x_1'$ literal is necessary in a realization of $T$.

In general, $T_{1k}$ and $T_{0k}$ can be formed when the partitioned $p$x $(n+3)$ matrix is found from a specified $T$. By applying the cases of the residues test to $T_{1k}$ and $T_{0k}$, the unnecessary literals, if any, can be found.

A compact and precise expression for representing the $T$ given in example 1 is $\Sigma$ *(2,6,13,15)*. That is, $T$ is specified by the sequence of decimal integers for those rows of the matrix of combinations for which $T = 1$. This is called the standard sum form of $T$.

Example 2:

$$T\ (x_3\ x_2,\ x_1,\ x_0) \equiv \Sigma\ (14) \equiv x_3\ x_2\ x_1\ x_0'$$

At first sight, the latter procedure may appear quite lengthy. However, this is not actually the case. The length is due to two causes: 1) an introduction to the fundamental theory with its nomenclature, mechanics and many details which with a little practice can become automatic; and 2) lack of a quick knowledge regarding whether a *0* or a *1* occurs in a particular row of the $p$x $(n+3)$ matrix corresponding to the $x_k$ column.

Now a procedure will be developed which circumvents the need to use the binary representation of $T$ and which allows the direct results from the decimal specification of $T$ alone; and in fact, does away with the need for a conversion table from decimal to binary numbers in order to effect the location of the *1*'s and *0*'s when given the decimal number.

### DECIMAL PROCEDURES

Consider a Boolean function $T$ of $(n+1)$ variables, expressed by $T \equiv \Sigma\ (D)$, where $D$ is the set of decimal integers representing the rows of the matrix of combinations for which $T = 1$.

The decimal integer $d$ will be said to contain $2^k$ in its binary number representation if $b_k = 1$ in the following identity:

$$d = \sum_{k=1}^{n} b_k\ 2^k$$

If $a$ is a real number let [a] denote the greatest integer not exceeding $a$. Under the above conditions:

*lemma*: The decimal integer $d$ contains $2^k$ in its binary number representation if and only if $[d \cdot 2^{-k}]$ is odd. The proof of the lemma is given in appendix I.

Example 3:

Let us see if 78 contains 4 in its binary number representation.

$$\left[\frac{78}{2^2}\right] \equiv [19\tfrac{1}{2}] = 19 \text{ which is odd.}$$

Therefore 78 does contain 4 in its binary number representation. In fact, 78 is 1 0 0 1 1 1 0 in binary form, where the 1 in the third position from the right indicates the presence of a 4.

Example 4:

Let us see if 105 contains 4 in its binary representation.

$$\left[\frac{105}{4}\right] \equiv [26\tfrac{1}{4}] = 26 \text{ which is even.}$$

Therefore 105 does not contain 4 in its binary representation. In fact, 105 is 1 1 0 1 0 0 1 in binary form, where the 0 in the third position from the right indicates the absence of a 4.

By means of the above lemma, it is possible to rapidly determine whether or not a given integer contains $2^k$ in its binary representation. It is convenient to define $p_k = [d \cdot 2^{-k}]$ as the placement quotient $(pq)$ of $d$ with respect to $2^k$, where $d \epsilon D$ *in* $T = \Sigma\ (D)$.

The decimal procedure for performing a residue test is the following:

(A) Let $D'$ denote the subset of $D$ with odd $pq$'s. Associate $x_k$ with the set $D'$. Let $D''$ denote the subset of $D$ with even $pq$'s. Associate $x_k$ with the Set $D''$.

(B) Add $2^k$ to each member of $D''$. Call this set $\bar{D}$. (Equivalently subtract $2^k$ from each member of $D'$. Call this set $\hat{D}$.)

(C) Examine $D'$ and $\hat{D}$ for inclusion relations. (Equivalently examine $D''$ and $\hat{D}$ for inclusion relations). By the above procedure, $T$ has been put in the form of the identity given by (3).

(D) If
$$D' \supset \bar{D}$$
or $\hat{D} \supset D''$
then $x_k'$ is not necessary in a realization of $T$. If
$$\bar{D} \supset D'$$
or
$$D'' \supset \hat{D}$$
then $x_k$ is not necessary in a realization of $T$.

Example 5: (See below).

Check the leading and trailing variables ($x_4$ and $x_0$, respectively, in this example) first, since they are tested almost by inspection.

Example 5:

$T\ (x_4, x_3, x_2, x_1, x_0) \equiv \Sigma\ (2, 3, 4, 5, 6, 7, 12, 13, 22, 23, 30, 31)$

where, of course, $D \equiv 2, 3, 4, 5, 6, 7, 12, 13, 22, 23, 30, 31$

(1) Checking $x_4^*$ first:

$$\left[\frac{d}{{}_2k}\right] \equiv \left[\frac{d}{{}_2 4}\right] \equiv \left[\frac{d}{16}\right],\ d \epsilon D.$$

each $d < 16$ gives $\left[\frac{d}{16}\right] = 0$, even $pq$

each $d \geqq 16$ gives $\left[\frac{d}{16}\right] = 1$, *odd* $pq$.

hence $D' = 22, 23, 30, 31$
and $D'' = 2, 3, 4, 5, 6, 7, 12, 13$
also $\bar{D} = 18, 19, 20, 21, 22, 23, 28, 29$

and $\hat{D} = 6, 7, 14, 15$

Since $D' \not\supset \bar{D}$ and $\bar{D} \not\supset D'$, Case 3 of the residue test holds. Therefore both $x_4$ and $x_4'$ are necessary in a realization of $T$. (Comparing $D''$ and $\hat{D}$ one gets the same results).

(2) Checking $x_0^*$ next:

$$\left[\frac{d}{{}_2k}\right] \equiv \left[\frac{d}{{}_2 0}\right] = d,\ d \epsilon D.$$

Hence $D' = 3, 5, 7, 13, 23, 31$
and $D'' = 2, 4, 6, 12, 22, 30$
also $\bar{D} = 3, 5, 7, 12, 23, 31$
and $\hat{D} = 2, 4, 6, 12, 22, 30$

Since $D' \equiv \bar{D}$, Case 1 of the residue test holds and neither $x_0$ nor $x_0'$ are necessary in a realization of $T$. (Comparing $D''$ and $\hat{D}$ one gets the same results).

(3) Checking $x_3^*$ next:

$$\left[\frac{d}{{}_2k}\right] \equiv \left[\frac{d}{{}_2 3}\right] \equiv \left[\frac{d}{8}\right], d \epsilon D.$$

Hence $D' = 12, 13, 30, 31$
and $D'' = 2, 3, 4, 5, 6, 7, 22, 23$
also $\bar{D} = 10, 11, 12, 13, 14, 15, 30, 31$
and $\hat{D} = 4, 5, 22, 23$

Since only $\bar{D} \supset D'$, Case 2 of the residue test holds. Since $D'$ is associated with $x_3$, $x_3$ is not necessary but $x_3'$ is necessary in a realization of $T$.

(4) Checking $x_2^*$ next:

$$\left[\frac{d}{{}_2k}\right] \equiv \left[\frac{d}{{}_2 2}\right] \equiv \left[\frac{d}{4}\right],\ d \epsilon D.$$

Hence $D' = 4, 5, 6, 7, 12, 13, 22, 23, 30, 31$
and $D'' = 2, 3$
also $\bar{D} = 6, 7$
and $\hat{D} = 0, 1, 2, 3, 8, 9, 18, 19, 26, 27$

Since $D' \supset \bar{D}$, Case 2 of the residue test holds. Since $\bar{D}$ is associated with $x_2'$, $x_2'$ is not necessary but $x_2$ is necessary in a realization of $T$.

(5) Checking $x_1^*$ next:

$$\left[\frac{d}{{}_2k}\right] \equiv \left[\frac{d}{{}_2 1}\right] \equiv \left[\frac{d}{2}\right],\ d \epsilon D.$$

Hence $D' = 2, 3, 6, 7, 22, 23, 30, 31$
and $D'' = 4, 5, 12, 13$
also $\bar{D} = 6, 7, 14, 15$
and $\hat{D} = 0, 1, 4, 5, 20, 21, 28, 29$

Since $D' \not\supset \bar{D}$ and $\bar{D} \not\supset D'$, case 3 of the residue test holds and both $x_1$ and $x_1'$ are necessary in a realization of $T$.

## Appendix I

Theo: $P_k \equiv C_k$ (mod B), $k = 1, 2, \ldots n$

Proof: $d = C_n B^n + \ldots + C_{k+1} B^{k+1} + C_k B^k + C_{k-1} B^{k-1} + \ldots + C_0$

$$\frac{d}{B^k} = \sum_{i=1}^{n-k} C_{i+k} B^i + C_k + \sum_{i=1}^{k} C_{k-i} B^{-i}$$

Put $f_k = \sum_{i=1}^{k} C_{k-i} B^{-i}$; *since* max $C_{k-i} = B-1$, then

$$0 \leq f_k \leq (B-1) \sum_{i=1}^{k} B^{-i}$$

Hence for every finite $k$

$$0 \leq f_k < (B-1) \sum_{i-1}^{\infty} B^{-i} = (B-1) \frac{\frac{1}{B}}{1-\frac{1}{B}} = 1$$

Putting $S_k = \sum_{i=1}^{n-k} C_{i,k} B^i$ notice that

$S_k = B \cdot S_{k+1}$, hence

$$P_k \equiv [\frac{d}{B^k}] \equiv S_k + C_k \equiv B\, S_{k+1} + C_k$$

Therefore $P_k \equiv C_k \pmod{B}$.

## Appendix II

If $T$ is expressed in standard product form (in terms of the zeros of transmission) as

$$T = \pi (D),$$

where $D$ is the set of decimal integers representing the rows of the matrix of combinations for which $T = 0$, it is only necessary to interchange the association of $x_k$ and $x_k'$ relative to $D'$ and $D''$ to effect a residue test. The inclusion relations then follow as before.

## Summary

For reasons of economy it is advantageous to synthesize switching circuits with as few components as possible. This paper develops a decimal procedure which serves as a necessary condition for realizing a minimum-component circuit. Thus, if a circuit is found, with the specified logical properties, using only the number of components defined by the procedure, it is known that one could not realize any circuit with the specified logical properties using fewer components. In this sense, a lower bound to the number of components needed in the realization of a circuit with specified logical properties is established.

## Literature Cited

Birkhoff, G., and S. MacLane. 1953. A survey of modern algebra. New York, Macmillan Co., 472 pp.

Quine, W. V. O. 1952. The problem of simplifying truth functions. Amer. Math. Monthly, 59:521-531.

Shannon, C. E. 1949. The synthesis of two-terminal switching circuits. Bell System Tech. Jour., 28:59-98.