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CONSERVATION OF ENERGY IN AN INELASTIC COLLISION

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Mechanics teaches that for collisions between real bodies, momentum is always conserved and some linear kinetic energy is always lost. In considering this contrast between the vector quantity (momentum) and the scalar quantity (kinetic energy), I have had occasion to solve the following problem in classical mechanics.

A block of mass *m* slides on a frictionless plane with initial velocity v_o. It collides with a second block of mass *M*, which is initially at rest and which is provided with a spring bumper (fig. 1). After the "collision," the mass *m* is assumed to attach itself to the end of the spring, and the problem is to find the equations of motion for the blocks and to resolve the various energy terms.

Using the symbols indicated in figure 1 and assuming that at time t=O, the initial conditions are

$$= v_o,\ V_y = V_o = O,\quad x_o = O,\ y_o = D_o,\ V_x$$

the differential equations of motion will be

$$mD^2x = -kD_o + k(y - x) \qquad (1)$$

$$MD^2y = +kD_o - k(y - x), \qquad (2)$$

where $D \equiv \frac{d}{dt}$.

Elimination of x (or y) between equations 1 and 2 leads to the following single equation in y (or x):

$$(D^4 + \phi^2 D^2)\, y = O, \qquad (3)$$

where $\phi^2 = \frac{k}{\mu}$ and $\mu = \frac{mM}{m+M}$ = reduced mass.

The solution of equation 3 by standard means results in the following expressions for x and y, the instantaneous positions of the two masses after the collision:

$$x = \frac{\mu}{m\phi} v_o \sin \phi\, t + \frac{\mu}{M} v_o t \qquad (4)$$

$$y = \frac{-\mu}{M\phi} v_o \sin \phi t + \frac{\mu}{M} v_o t + D_o \qquad (5)$$

For an ordinary inelastic collision the relationship between the initial and final velocities would be

$$m\, v_o = (m+M)\, \overline{V}$$

$$\overline{V} = \left(\frac{m}{m+M}\right) v_o = \frac{\mu}{M} v_o = \text{final}$$

velocity. Hence equations 4 and 5 would take the form

$$x = \frac{M}{m}\,\frac{\overline{V}}{\phi}\sin\phi t + \overline{V}t = \frac{v_0-\overline{V}}{\phi}$$

$$\sin\phi t + \overline{V}t \qquad (4')$$

$$y = \frac{-\overline{V}}{\phi}\sin\phi t + \overline{V}t + D_0 \qquad (5')$$

The solutions consist of a constant velocity part and an oscillatory part.

For the energy calculations the expressions for the instantaneous velocity of the masses are required:

$$\dot{x} = (v_0 - \overline{V})\cos\phi t + \overline{V} \qquad (6)$$

$$\dot{y} = \overline{V}\,(1 - \cos\phi t) \qquad (7)$$

It is interesting to note that the mass which was originally at rest periodically comes to rest *after* the collision. This is true regardless of the relative values of the masses.

With the use of equations 6 and 7 and by taking time averages, it may now be shown that

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\overline{V}^2 + \frac{1}{2}M\overline{V}^2 + \frac{1}{4}M\overline{V}^2 +$$

$$\frac{1}{4}m(v_0-\overline{V})^2 + \frac{1}{4}\mu v_0^2 \qquad (8)$$

The term on the left-hand side of the equation is the initial kinetic energy of the mass m. The first term on the right-hand side is the "steady-state" kinetic energy of mass m after the collision, and the second term is the corresponding kinetic energy for mass M. The next two terms are the average values of the *vibrational* kinetic energies of the masses, and the last term is the average value of the *potential* energy of the spring after collision. All the initial energy is therefore completely accounted for by purely mechanical expressions.

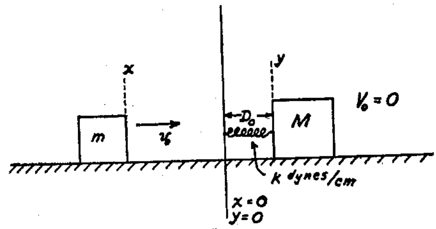

Fig. 1.—Collision of two masses on frictionless plane.

In equation 8 the last three "spring terms" are the energy terms which produce heat as the spring motion gradually dies out. Whether the heat production be accomplished in a shorter or longer length of time seems to be immaterial to the problem. Thus it appears that there is no such thing as an "inelastic" collision in nature, only *elastic* collisions for which the time of heat production may be extremely short. Actually a collision with heat losses is what we all mean when we speak of inelastic collisions, but it is well to bear in mind that the heat losses are due to the real properties of materials and not to any inherent tendency of collision processes to convert linear kinetic energy into heat.