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A Computer Approach to Investigating Inner Automorphisms of Cayley's Albegra

by

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1. The Algorithm. A computer was used to discover relationships between the inner automorphisms of Cayley's algebra C over the field of real numbers. Using the formulas discovered, the result that every automorphisms of C is an inner automorphism was proved. This paper explains the development of computer software used in finding a proof that every automorphism of C can be computer generated using only the standard operations of C. A hierarchy chart organizing the results used in the proof is given in figure 1. The crucial task performed by the computer was to find examples of explicit formulas expressing the reflection in any quaternion subalgebra H of C in a natural, computable form. A computer was used to search for examples of possible formulas. The formulas discovered are given in Theorem 1.2 below.

Let C denote Cayley's algebra defined over the field R of real numbers. In terms of a given basis $i_0 = 1, i_1, i_2, \ldots, i_7$, a general element $\xi = \sum_{s=0}^{7} x_s i_s$ of C has real components x_s. The norm of ξ is given by $N(\xi) = \sum_{s=0}^{7} x_s^2$ and the real part by $R(\xi) = x_0$. The element $\bar{\xi} = 2x_0 - \xi$ of C is called the conjugate of ξ. Any element ξ of C satisfies the rank equation

$$\xi^2 - 2R(\xi)\,\xi + N(\xi) = 0. \qquad (1)$$

Multiplication in $\underline{C}$ is given by the formula

$$\xi\eta = \xi_0\eta_0 - \bar{\eta}_1\xi_1 + (\eta_1\xi_0 + \xi_1\bar{\eta}_0)\, i_4 \tag{2}$$

where ξ_t, η_t $(t = 0,1)$ are elements of the quaternion subalgebra $\underline{H}$ of $\underline{C}$ generated by $\{i_s\}_0^3$ over $\underline{R}$. Programming multiplication of Cayley numbers is similar in difficulty to programming multiplication for matrices with eight rows and eight columns. A sample program is given below of programming Cayley multiplication using matrix multiplication.

An element ξ of $\underline{C}$ has a unique multiplicative inverse $\bar{\xi}\ /\ N(\xi)$ if and only if $N(\xi)$ is non-zero. The following theorem is proved in [3].

<u>Theorem 1.1.</u> A non-real element σ of $\underline{C}$ induces an inner automorphism

$$[\sigma] : \xi \to \sigma\xi\sigma^{-1}$$

of $\underline{C}$ if and only if

$$4\, R^2(\sigma) - N(\sigma) \neq 0.$$

An automorphism τ is called a reflection if $\tau^2 = 1$ and $\tau \neq 1$. A reflection τ is the identity mapping 1 on some quaternion subalgebra of $\underline{C}$ and -1 on the orthogonal complement. Jacobson [2] has proved that every automorphism of $\underline{C}$ is a product of reflections. The formulas for the reflection in quaternion subalgebra $\underline{H}$ discovered using the computer are given in the following theorem.

Theorem 1.2. Let $\sigma=\frac{1}{2}(1 + i_1 + i_2 + i_3)$ and $\rho=i_3\sigma$. Then

$$[(\sigma\rho)^{-1}][\rho\sigma] \text{ and } \{[\sigma][\rho^2]\}^2$$

both represent the reflection in H.

The theorem can be proved using the Dickson multiplication (2). The proof of the following theorem is given in [5].

Theorem 1.3. Every automorphism of C is a product of inner automorphisms.

2. Computer Implementation. Using the method of Cayley [1] to represent products, the following BASIC-PLUS program performs Cayley multiplication. The arrays A and B represent the coefficients of the Cayley numbers to be multiplied and the array M gives the answer.

```
 10  DIM A(8,1), B(1,8), C(8,8)
 20  DIM D(8,8), Z(8), M(1,8)
 30  MAT READ A
 40  MAT READ B
 50  MAT C=A*B
 60  S=S+C(1,1)
 70  FOR I = 2 TO 8
 80      S=S-C(I,I)
 90  NEXT I
100  FOR I = 2 TO 8
110      E(I) = C(1,I) + C(I,1)
120      FOR J = 2 TO 8
130          D(I,J) = C(I,J) - C(J,I)
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```
140    NEXT J
150  NEXT I
160  Z(1) = D(3,4) + D(5,6) + D(8,7) + E(2)
170  Z(2) = D(4,2) + D(5,7) + D(6,8) + E(3)
180  Z(3) = D(2,3) + D(7,6) + D(5,8) + E(4)
190  Z(4) = D(6,2) + D(7,3) + D(8,4) + E(5)
200  Z(5) = D(2,5) + D(4,7) + D(8,3) + E(6)
210  Z(6) = D(3,5) + D(6,4) + D(2,8) + E(7)
220  Z(7) = D(3,6) + D(4,5) + D(7,2) + E(8)
230  M(1,1) = S
240  M(1,K) = Z(K-1)  FOR K = 2 TO 8
250  MAT PRINT M
260  DATA --- coefficients of first Cayley number
270  DATA --- coefficients of second Cayley number
280  END
```

By nesting the above program suitably, repeated applications produce the images of the basic units of C under automorphisms of the form given in Theorem 1.1. For example, $\sigma = \frac{1}{2}(1 + i_2 + i_4 + i_7)$ gives the following automorphism.

$$
\begin{aligned}
i_0 &\to i_0 = 1 \\
i_1 &\to -\tfrac{1}{2}(i_1 + i_3 + i_5 + i_6) \\
i_2 &\to \tfrac{1}{2}(i_4 + i_5 - i_6 + i_7) \\
i_3 &\to \tfrac{1}{2}(i_1 - i_3 + i_4 - i_7) \\
i_4 &\to \tfrac{1}{2}(i_2 - i_3 + i_6 + i_7) \\
i_5 &\to \tfrac{1}{2}(i_1 - i_2 - i_5 + i_7) \\
i_6 &\to \tfrac{1}{2}(i_1 + i_2 - i_4 - i_6) \\
i_7 &\to \tfrac{1}{2}(i_2 + i_3 + i_4 - i_5)
\end{aligned}
$$

Suitable choices of chains of automorphism generating elements yielded Theorem 1.2. Programs were written for a PDP-11/40 using MICS/11 operating system under the RSTS/E time sharing monitor in BASIC-PLUS.

REFERENCES

1. A. Cayley, On the eight-square imaginaries, American Journal of Mathematics 4 (1881), 293-296. Collected Mathematical Papers 11, 368-371.
2. N. Jacobson, Composition algebras and their automorphisms, Rend. Circ. Mat. Palermo 7 (1958), 55-80.
3. P. J. C. Lamont, Arithmetics in Cayley's algebra, Proc. Glasgow Math. Assoc. 6 (1963), 99-106.
4. P. J. C. Lamont, Factorization and arithmetic functions for orders in composition algebras, Glasgow Math, J. 14 (1973), 86-95.
5. P. J. C. Lamont, Computer generated natural inner automorphisms of Cayley's algebra, Glasgow Math. J. 23 (1982), 187-189.

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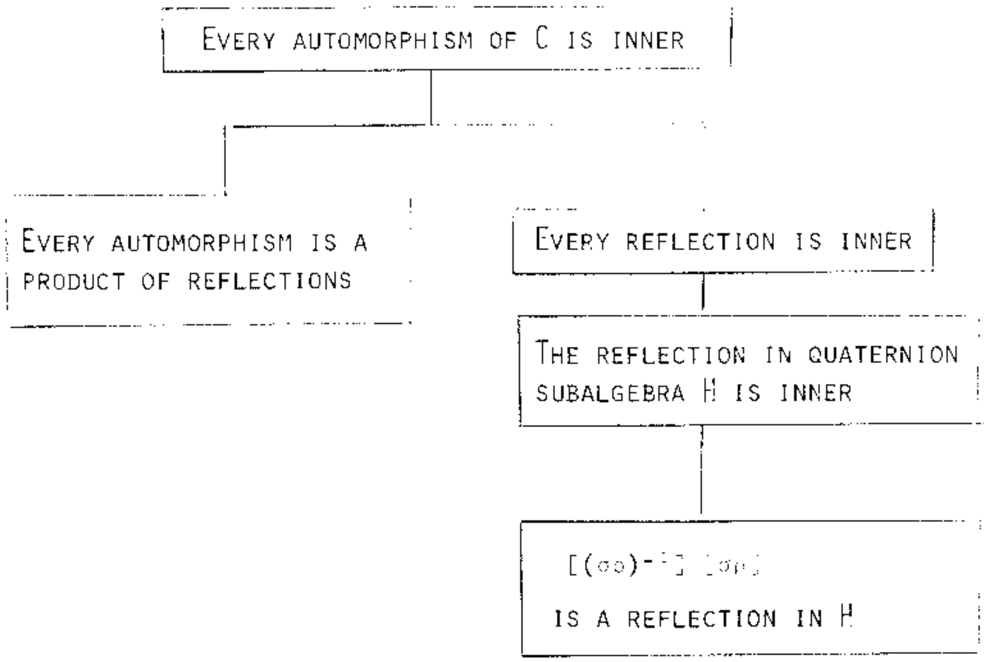
EVERY AUTOMORPHISM OF C IS INNER
EVERY AUTOMORPHISM IS A PRODUCT OF REFLECTIONS
EVERY REFLECTION IS INNER
THE REFLECTION IN QUATERNION SUBALGEBRA H IS INNER
$[(\sigma\sigma)^{-1}]$ $[\sigma\mu]$
IS A REFLECTION IN H

Figure 1.