MINI-MAX APPROXIMATIONS OF THE DIRICHLET PROBLEM FOR A CLASS OF SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: Let W be a holomorphic solution of the equation

$$\partial^2 W/\partial z\partial\bar{z} = A(|z|^2)|z|\partial W/\partial|z|, \ |z|<1$$

that is continuous on $|z|\leq 1$. And let the first n-1 tangential derivatives of W be continuous with the (n-1)st. satisfying a first order Hölder condition. Then determine the mini-max error β_{nm} and the extremal solution for the best "polynomial" approximation of degree m on $|z|\leq 1$, which leads to an extension of the classical theorem of Favard, Achieser and Krein in analytic function theory.

INTRODUCTION

Analytic functions U, on the disk $\Delta:|z|<1$, viewed as harmonic functions, are the "holomorphic" solutions of the equation

$$\partial^2 U/\partial z\partial\bar{z} = 0, \ z\in\Delta. \tag{1}$$

A natural extension of an analytic function on a disk is given by a solution of the elliptic equation

$$L(W) := \partial^2 W/\partial z\partial\bar{z} - A(|z|^2)|z| \ \partial W/\partial|z| = 0, \ z\in\Delta \tag{2}$$

whose admissible coefficients A are real-valued analytic functions on the closure of Δ. Analytic solutions W of eqn. (2) arise when extending univalent function theory [9]. Under suitable coordinate transformations they reduce to conically symmetric potentials [2] which are the radially independent harmonic functions in E^3; these also occur in problems in rheology [8,10] and wave mechanics [2].

The classical theorem of *Favard, Achieser* and *Krein* [1,4] as well as the "analytic" version by *Babenko* [4,6] relate the smoothness of the boundary values of a holomorphic function U to the error in the best polynomial approximation of U on the $\partial\Delta$; hence, globally on Δ by the maximum principal. This paper extends these classical results to holomorphic solutions W of eqn. (2). To accomplish this, we apply some properties of holomorphic representations of W developed by Rusheweyh [9] in the study of the univalent function theory of W.

THE DEFINITIONS AND FORMULAE

It follows from I. N. Vekua [10] that a complete set for analytic solutions on Δ relative to the uniform norm, $||W||_{L^\infty(\Delta)} = \sup\{|W(z)| : z\varepsilon\Delta\}$, is given by the functions

(3) $$W_k(z) = e^{ik\phi}d_{|k|}(r),\ k = 0, \pm 1, \pm 2, \ldots,\ z = re^{i\phi}$$

where the functions $d_k(r)$ are the regular solutions of the following two-point boundary value problem

$$[d^2/dr^2 + \{r^{-1} - 4rA(r^2)\}d/dr - k^2r^{-2}]d_k = 0,$$

$$d_k(0) = d_k(1)-1 = 0,\ k = 0,1,2,\ldots .$$

A complex-valued function W that is twice continuously differentiable on Δ is then an *analytic solution* of eqn. (2) if and only if there exists a sequence of complex constants $\{c_k\}_{k=-\infty}^{\infty}$ such that the expansion

(4) $$W(z) = \sum_{k=-\infty}^{\infty} c_k e^{ik\phi}d_{|k|}(r)$$

is absolutely and uniformly convergent on compacta of Δ. The *holomorphic solutions* are those analytic solutions that expand as

(5) $$W(z) = \sum_{k=0}^{\infty} c_k e^{ik\phi}d_k(r),\ z = re^{i\phi}\varepsilon\Delta.$$

These may be defined as the $\star$-convolution of a holomorphic associate

(6) $$U(z) = \sum_{k=0}^{\infty} c_k z^k,\ z\varepsilon\Delta$$

with the Poisson kernel

(7) $$P_\rho(z) = \sum_{k=0}^{\infty} d_k(\rho)z^k,\ z\varepsilon\Delta,\ 0\leq\rho<1.$$

via the "Hadamard product"

(8) $$W(z) = \tau_A U(re^{i\phi}) = P_r(z)\star U(e^{i\phi}),\ z\varepsilon\Delta$$

The invertible operator τ_A associates a unique holomorphic U with each holomorphic W.

THE BEST POLYNOMIAL APPROXIMATION

Let $\mathcal{D}_n$ designate the set of all holomorphic solutions to Dirichlet problems of the class:

(9) $$L(W) = 0, z\varepsilon\Delta$$
$$W = F, z\varepsilon\partial\Delta$$

where W is continuous on the closure of Δ, $cl(\Delta)=\Delta\cup\partial\Delta$; and, in addition

$W\varepsilon\{W:D_\theta^{n-1}W|_{\partial\Delta}$ meets a first order Hölder condition$\}\bigcap \bigcup_{j=1}^{n-1}\{W:D_\theta^j\ W\ \varepsilon\ C(\partial\Delta)\}$.
The specified Hölder condition is

$$||D_\theta^{n-1}W(1,\theta') - D_\theta^{n-1}W(1,\theta'')||_{L^\infty[0,2\pi)} < K\ ||\theta' - \theta''||_{L^\infty[0,2\pi)}$$

for all θ', θ'' in $[0,2\pi)$ and $K > 0$.

The fundamental sets of best approximation are formed from the analytic polynomial solutions of eqn. (2) by

$$(10)\quad T_m = \{S:S(re^{i\phi}) = \sum_{k=-m}^{m} c_k e^{ik\phi} d_{|k|}(r),\ \{c_{-m},\ldots,\ c_m\}\subset C\}, m = 0,1,2,\ldots\ .$$

The first objective is to develop from these sets, a generalization of the theorem of Favard, Achieser and Krein [4] which is itself a generalization of the famous theorem of *D. Jackson* [3] on polynomial approximation. This we accomplish in

THEOREM 1. The mini-max error in the approximation of the class of Dirichlet problems D_n over the set T_m of analytic polynomials of degree m is given by

$$(11)\qquad \beta_{nm} = \text{Sup}\ \{\inf\{||W-S||_{L^\infty(\partial\Delta)} : S\epsilon T_m\} : W\epsilon D_n\}$$

where $\beta_{nm} = K_n/(m+1)^n$ and

$$(12)\qquad K_n = \frac{4}{\pi}\sum_{j=0}^{\infty} (-1)^{j(n+1)}/(2j+1)^{n+1}$$

Furthermore, the extremal function $W_0^{(n)}\varepsilon D_n$ is of the form

(i) n an odd integer

$$W_0^{(n)}(re^{i\phi}) = \frac{2}{\pi}\sum_{k=1}^{\infty}\left[\frac{(-1)^k-1}{k}\right]\sin[(m+1)k\phi]d_k(r)$$

and

(ii) n an even integer

$$W_0^{(n)}(re^{i\phi}) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\cos[(2m+1)k\phi/3]d_k(r).$$

If $U_0^{(n)}$ is any other extremal solution, then

$$U_0^{(n)}(re^{i\phi}) = \lambda W_0^{(n)}(re^{i(\phi-\phi_0)}) + c \text{ where } \lambda = \pm 1,$$

$\phi_0\varepsilon[0,2\pi)$ and c is a constant.

Proof: The map τ_A provides a natural isomorphism from the linear space of holomorphic solutions U of the Dirichlet problems d_n:

$$\partial^2 U/\partial z\partial\overline{z} = 0, \; z\varepsilon\Delta$$
$$U = F, \; z\varepsilon\partial\Delta$$

$$U\varepsilon C(cl(\Delta))\cap\{U : D_\theta^{n-1}U|_{\partial\Delta}, \text{ first order Hölder}\}\cap\bigcup_{j=1}^{n-1}\{U{:}D_\theta^j U\varepsilon C(\partial\Delta)\}$$

and the (extended) linear space $\mathcal{D}_n$.

Therefore, let us define the set of polynomials

$$(13) \qquad t_m = \{s{:}s(z) = \sum_{k=-m}^{m} c_k z^k,\ \{c_{-m},\dots,c_m\}\subset C\},\ m = 0,1,2,\dots\ .$$

The theorem of Favard, Achieser and Krein established that (1) relative to d_n:

$$(14) \qquad \beta_{nm} = \text{Sup}\{\inf\{||V-s||_{L^\infty(\partial\Delta)} : s\varepsilon t_m\} : U\varepsilon d_n\}$$

where β_{nm} is evaluated above and that (2) the extremal solutions are:

(i) $U_0^{(n)}(1,\theta) = F^{(n)}(\theta) = \text{Sgn}[\cos(m+1)\theta]$, n-even

and

(ii) $U_0^{(n)}(1,\theta) = F^{(n)}(\theta) = \text{Sgn}[\sin(m+1)\theta]$, n-odd.

Any other extremal solution of eqn. (14) has the form $U_0^{(n)}(1,\theta) = \lambda F^{(n)}(\theta-\theta_0)+c$ where $\lambda = \pm 1$, $\theta_0\varepsilon[0,2\pi)$ and c is a constant.

By the ordinary maximum principal for holomorphic functions $U\varepsilon d_n$ it follows that

$$(15) \qquad \beta_{nm} \le \inf_{s\varepsilon t_m}||U-s||_{L^\infty[0,2\pi)} \le \inf_{s\varepsilon t_m}||U(\rho e^{i\theta})-s(\rho e^{i\theta})||_{L^\infty[0,2\pi)}$$

$\rho<1$ with

$$(16) \qquad \limsup_{\rho\uparrow 1}\ \sup_{d_n}\ \inf_{t_m} ||U(\rho e^{i\theta}) - s(\rho e^{i\theta})||_{L^\infty[0,2\pi)} = \beta_{nm}$$

by the theorem of Favard, Achieser and Krein. Moreover, the *Hopf maximum principal* [2,9] shows that

$$(17) \qquad \inf_{S\varepsilon T_m} ||W-S||_{L^\infty[0,2\pi)} \le \inf_{S\varepsilon T_m} ||W(\rho e^{i\theta}) - S(\rho e^{i\theta})||_{L^\infty[0,2\pi)}$$

To compare the values of $\tau_A U$ and W on $\partial\Delta$, note first the limit $\tau_A s(\rho e^{i\phi}) \to S(e^{i\phi})$ as $\rho\uparrow 1$ is uniform in $\phi\varepsilon[0,2\pi)$ so that by the Hopf maximum principal

(18) $\tau_A s(z) = S(z),\ s\epsilon t_m,\ z\epsilon cl(\Delta).$

Now to show that the limit $\tau_A U(\rho e^{i\phi}) \to W(e^{i\phi})$ as $\rho \uparrow 1$ is uniform in $\phi\epsilon[0,2\pi)$, we observe as in [9] that the (Poisson) kernel of τ_A is positive on Δ. Then by an application of the Hopf maximum principal and *Korovkin's* [3,7,9] famous theorem on positive operators, when the holomorphic function $U\epsilon d_n$ extends continuously to $cl(\Delta)$, the transform $W = \tau_A U$ extends continuous to $cl(\Delta)$. Therefore, the identity

$$(19)\qquad \sup_{\mathcal{D}_n}\inf_{T_m}||W-S||_{L^\infty(\partial\Delta)} = \sup_{d_n}\inf_{t_m}||U-s||_{L^\infty(\partial\Delta)}$$

follows as we have just shown that

$$(20)\qquad ||W-S||_{L^\infty(\partial\Delta)} = ||U-s||_{L^\infty(\partial\Delta)},\ W-S = \tau_A(U-s).$$

In other words the isomorphism $\tau_A: d_n \to \mathcal{D}_n$ is an isometry, too. The form of the extremal function $W_0^{(n)}$ is found by expanding the τ_A-associated extremal functions $F^{(n)}(\theta)$ with the coefficients identified according to eqn. (5) completing the proof.

We summarize this theorem as follows: the holomorphic solutions of eqn. (9), which are continuous on the $cl(\Delta)$ and have continuous (n-1)st tangential derivatives along with the n-th tangential derivative bounded by 1, can be approximated by an error of no less than β_{nm}.

The "analytic" version is the generalization of Babenko's theorem [4] which replaces the analytic polynomial approximation by the holomorphic polynomial set

$$(21)\qquad V_m = \{p : p(re^{i\phi}) = \sum_{k=0}^{m} c_k e^{ik\phi} d_k(r),\ \{c_0,\dots,c_m\}\subset C\}.$$

$m = 0,1,\dots$. To study this properly, let H_n designate the holomorphic solutions of $\mathcal{D}_n$ whose n-th tangential derivative is bounded in modulus by one on the $\partial\Delta$. If we apply reasoning from the first theorem to Babenko's theorem, we immediately progress to

<u>THEOREM 2.</u> The mini-max error in the approximation of the class of Dirichlet problems H_n over the set V_m of holomorphic polynomials of degree m is given by

$$(22)\qquad \operatorname{Sup}\{\inf\{||W-P||_{L^\infty(\partial\Delta)} : P\epsilon V_m\} : W\epsilon H_n\} = \Gamma(m-n)/\Gamma(m)\quad m \geq n.$$

The holomorphic extremal solutions have the form

$$U_0^{(n)}(z) = cW_{m+1}(z) + \sum_{k=0}^{n-1} c_k W_k(z)$$

where c is a constant.

CONCLUDING REMARKS

A variety of elliptic partial differential equations that are found in applications [see 5,8] have solutions that may be represented as (usually integral) transforms of associated analytic or harmonic functions of a single complex-variable. Moreover, on certain regions with Liapunov surfaces for boundaries (in our problem $|z| = 1$) the associate and the solution are identical. Hence, approximation of the associate leads by a maximum principal to global approximation of the solution. This suggests other extensions of the theorem of Favard, Achieser and Krein.

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