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Three Paradoxes and the Class Operator Rule

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ABSTRACT

The Greeling and village barber paradoxes are resolved by plausibly correcting their propositions for ambiguity. Resolution of the Russell paradox requires two more corrections: (1) adding "all and only" to Russell's definition of his class W and (2) shifting attention from classes to class names as class members. All of the propositions are then analogs of a postulated general proposition Z that defines class S whose member O performs definable operation P upon all and only those members of class S that are notself P-operated-upon. The three propositions are linked by an important property of member O, namely, logical inability to receive its own operation. That property is the basis of the class operator rule (COR). The COR replaces the propositional scheme P(x) of class formation, thus eliminating Zermelo's hereditary classes, removing the ad hoc distinction between set classes (sets) and proper classes (nonsets), and restoring Frege's Axiom of Abstraction to independent status. Class membership is restricted to class names. The word "class" is defined in terms of two kinds of class names: contained and uncontained as members of named classes. Frege agreed with Russell's criticism of Frege's Rule 5, but he disagreed with Russell's discovery of contradiction in notself-predicated predicates. Frege knew that the fundamental rule of symbolic language requires all predicates in logical propositions to be notself-predicated; Russell didn't.

INTRODUCTION

Fraenkel and Bar-Hillel (1958) and Quine (1962) discuss paradoxes, noting that discovery of a paradox occasionally leads to an important alteration in the foundations of logic. Quine describes a paradox in general as any conclusion that at first seems absurd but that has an argument to support it. He divides paradoxes into (1) those whose supporting arguments include a discernible flaw and (2) those whose arguments appear to be flawless. He refers to the first group as paradoxes, to the second group as antinomies.

Chief purpose of this paper is to identify the foundations of three structurally similar paradoxes, those named after Bertrand Russell, Kurt Grelling and the village barber. Knowledge of the foundations is sought in the hope that it will lead to objective resolutions of the paradoxes. Another purpose is to point out a significant difference between Russell's paradox and the other two. Additional purposes are to present the class operator rule and to offer a definition of the term "class."

METHOD

This paper defends the following conclusions:

1 - Class membership is restricted to class names.

2 - In the village barber scenario the barber can't be shaved, i.e., he remains permanently bearded.

3 - In the Greeling scenario the coined adjective "heterological" can't be Greeling-classified as either heterological or autological.

4 - In the Russell scenario, class W can't be contained in any class, including itself.

The barber, coined adjective "heterological" and class W are herein regarded as class operators with similar functions. Each operator performs a definable operation P upon all and only those members of a general class S which are notself P-operated-upon, class S being defined by the following proposition:

> Proposition Z -- In well-defined class S there is member O, which performs definable operation P upon all and only those members of class S which are notself P-operated-upon.

The listed four conclusions and Proposition Z lead to the class operator rule (COR), which is stated as follows:

> In any definition of class membership resulting from action by, or function of, an operator as defined in Proposition Z, no language shall be used that *excludes* the operator from its universal class or *includes* the operator in the notself P-operated-upon subclass of that universal class.

The COR is a generalization of the logical inability of every class operator to receive its own operation.

According to the barber paradox proposition, the barber shaves all and only those resident men in a particular village who do not shave themselves. The clause "men who do not shave themselves" is ambiguous; it refers to (a) men who are shaved but not by themselves and to (b) men who are never, never shaved, i.e., permanently bearded men. Ambiguity is avoided by replacing "men who do not shave themselves" with unambiguous "men who are notself shaved."

As explained later in this paper, the barber with unambiguous assignment will remain unshaved, i.e., permanently bearded. Argument leading to that conclusion relies upon ideas of monopoly and rational behavior of monopolists. Analogous conclusions that coined adjective "heterological" can't be grelling classified and that class W can't be contained in any class (including itself) also rely upon the monopoly argument.

Each of the four listed conclusions is based on a chain of reasoning that begins with recognition of ambiguity in Russell's definition of class W and in statements of the classical barber, Grelling and Russell paradoxes. The second link in that chain is plausible correction of the ambiguous language. The third and final link in the chain is derivation of logical conclusions from the plausibly corrected language.

RESULTS AND DISCUSSION

Classical Paradox Propositions

The propositions listed below are derived from Quine (1962). Although the wording differs from Quine it is regarded as equivalent by the author of this paper.

> Russell -- In the class of all and only classes there is class W that contains all and only those classes which do not contain themselves.
>
> Grelling -- In the class of all and only adjectives there is coined adjective "heterological" that grelling-classifies all and only the adjectives that do not grelling-classify themselves.
>
> Village Barber -- In the class of all and only men residing in a certain village there is a man barber who shaves all and only the men residing in that village who do not shave themselves.

The three propositions are clearly similar in structure. In each is a defined class, a defined member of that class and a definable operation P that the defined class member performs upon all and only the defined class members that are notself p-operated-upon.

The barber proposition in particular is examined carefully in the following paragraphs. Attention is given to interpreting the proposition and to literature arguments offered to sustain the alleged self-contradiction in the proposition.

Alleged Self-contradictory Conclusions from the Classical Paradox Propositions

Quine (1962) implies these conclusions: Class W contains itself if and only if class W does not contain itself. Coined adjective "heterological" grelling-classifies itself if and only if "heterological" does not grelling-classify itself. The barber shaves himself if and only if he does not shave himself.

Ambiguous Language in the Classical Paradox Propositions

Neither Fraenkel and Bar-Hillel (1958) nor Quine (1962) appear to recognize ambiguity in the clauses "classes that do not contain themselves," "adjectives that do not grelling-classify themselves," "village resident men who do not shave themselves." The Russell proposition requires class W to contain (a) notself contained classes (ordinary) and (b) uncontained classes. The Grelling proposition requires "heterological" to Grelling-classify adjectives that are (c) heterological (notself descriptive) and (d) neither heterological nor autological (self descriptive), e.g., "hungry." The barber proposition requires the barber to shave village resident men who are (e) notself shaved and (f) never, never shaved, i.e., permanently bearded. The propositions are ambiguous because of their dual and conflicting meanings. Communication clarity is improved by deleting the unreasonable alternative in each proposition (items b, d and f) as shown below.

Classical Paradox Propositions Corrected for Ambiguity

Russell -- In the class of all and only classes there is class W that contains all and only the notself contained classes.

Grelling -- In the class of all and only adjectives there is coined adjective "heterological" that grelling-classifies all and only the adjectives which are notself grelling-classified.

Village barber -- In the class of all and only men residing in a particular village there is a man barber who shaves all and only the men residing in that village who are notself shaved.

Logical Conclusions from Corrected Paradox Propositions

It will be shown first that the barber can't be self shaved and that he will refuse, on ethical grounds, to be notself shaved, thus remaining permanently bearded. Here is the argument. The word "only" in the definition-precision phrase "all and only" prevents the barber from being shelf shaved. The word "all" in that phrase gives the barber a monopoly on the notself shaving operation. If the barber allows himself to be notself shaved he becomes a joint-tort-feasor. Since the barber is presumed to be ethical and self respecting, he will avoid participating in any action that violates his shaving monopoly. Hence the word "all" in the "all and only" phrase bars the barber from being notself-shaved, thus requiring the barber to remain permanently bearded.

The same kind of monopoly argument applied to the Grelling and to the Russell propositions shows that "heterological" can't be grelling-classified as either heterological or autological, and that Russell's class W can't be contained as a member in any class, including itself. The monopoly argument may seem to be less persuasive in these two cases, but only because words and classes are inanimate objects.

Resolution of the Grelling and village barber paradoxes

Foundations of these two paradoxes are obviously ambiguous language in their propositions interpreted to require all adjectives to be grelling-classified and all men residing in a certain village to be shaved. Plausibly correcting the ambiguous language, as shown above, yields propositions that lead logically to the noted noncontradictory conclusions, thus resolving the paradoxes.

Important Difference between Russell's Paradox and the Others

Plausible elimination of ambiguity in the classical Russell paradox proposition, as presented above, fails to resolve the paradox. The unambiguous proposition remains self contradictory because it asserts that class W is a member of the class of all and only classes. That assertion conflicts with proof in later paragraphs of this paper (and mentioned in the METHOD section) that class W can't be a member of any class. Additional modification of the proposition, as shown later, is needed to resolve the paradox.

Proposition Z and General Class S
In well-defined multi-membered general class S there is member O, the class operator, which performs definable operation P upon all and only those members of class S which are notself P-operated-upon, said operation being consistent with the operating options of member O.

The plausibly corrected Grelling and barber paradox propositions are analogs of Proposition Z. For example, the class of all and only men residing in a certain village is an analog of class S; the barber is an analog of member O in class S; shaving is an analog of definable operation P; the barber's clients, all and only the men residing in the village who are notself P-operated-upon, comprise an analog of the members of class S which are notself P-operated-upon. Attention is called to an important feature of class S. The class S members which are notself P-operated-upon comprise a proper subclass of class S, and the minimum membership in the residual sub-class of class S is member O, the class operator.

If a plausible modification of the corrected Russell paradox proposition analogous to Proposition Z can be found, then all three paradox propositions can be linked by the class operator rule. Such proposition, or new definition of Russell's class W, is presented in the following paragraph.

New Definition of Russell's Class W
In the class of all and only class names residing in the domain of variable x in the propositional scheme P(x) of class formation there is "W", the name of Russell's class W, which denotes all and only those class names that are notself denoted in logical propositions.

The monopoly argument used earlier to show that the barber won't be shaved, that coined adjective "heterological" can't be grelling-classified, that class W can't be a class member works equally well to show that the name of class W can't be denoted in logical propositions.

The class names denoted by "W" comprise an analog of class S members that are notself P-operated-upon. Such class names comprise a proper subset of the class names in the domain of variable x in P(x) and "W" is the residual class name in that domain. "W" can't denote itself in logical propositions. The class operator rule and the fundamental convention of symbolic language both forbid self denotation. The result is consistent with rejection of self membered classes by current class theory.

Resolution of Russell's Paradox
The paradox is resolved by a three-step operation. One, plausibly correct the paradox proposition for ambiguity; two, add the definition-precision phrase "all and only" to Russell's definition of class W; three, shift attention from classes to class names as class members. The result is a definition of class W yielding uncontradictory conclusion that "W" can't be denoted in logical propositions. There is no contradiction in that conclusion, for if "W" were to be denoted in a logical proposition, that denotation must be accomplished with a *name* for "W" as the subject of the proposition. See Tarski (1944) and Linsky (1952).

Class operators and the Class Operator Rule (COR)

The village barber, coined adjective "heterological" and "W", the name of Russell's class W, are class operators. Each operator is an analog of member O in general class S as defined in Proposition Z, the latter first mentioned in the METHOD section of this paper. Logical inability of every class operator to receive its own operation is the foundation of the class operator rule (COR). The COR is stated as follows:

> In any definition of class membership based on action by, or function of, a class operator as defined in Proposition Z, no language shall be used that *excludes* the operator from its universal class or that *includes* the operator in the notself P-operated-upon proper subclass of that universal class.

Class membership

Earlier argument that correctly defined class W can't be a class member raises this question: "Is the inability of that class to be a class member restricted to that class or is such inability a feature common to all classes?" In his textbook on naive set theory Gleason (1966) throws light on the question. Quoting from Section 6 of Chapter 2: "When we wish to consider a set whose elements a, b, c, etc. can be explicitly listed, we may denote it by (a, b, c. . . . k). Formally, this notation can be used only when there are finitely many elements and we are prepared to write them out in full . . . . In connection with this notation, it must be emphasized that there is a distinction between an object x and the set (x) which has just that one object. Similarly, when a et A appears as an element of another set B, the elements of A are not counted among the elements of B, at least not by virtue of their membership in A. Suppose that A = (1,2) and B = (A, 1). Here 2 is an element of A but 2 is not an element of B. The fact that 1 is an element of A and 1 is an element of B is a coincidence.

The author of this paper interprets Gleason's quoted remarks about the list notation of elements to imply that membership of one class in another class is restricted to the name of the first class. That interpretation answers the question raised earlier. The provided inability of class W to be a class member is a feature common to all classes.

Russell's Rule for Collections: Self Membered Classes

In his study of class W, Russell (1908) arrived at this rule for collections: "Whatever involves all of a collection must not be one of the collection." Since the name of a collection involves, by denotation, all members of the collection, Russell's rule leads to conclusion that no class can contain its name as a member of the named class. The concept of self membered classes is invalidated by combining Russell's rule with the principle that class membership is restricted to class names.

Definition of the word "class"

A class is a linguistic association of one and only one uncontained class name with one or more contained class names, said uncontained name denoting collectively all and only said contained names, each contained name denoting a mathematical object that is a distinct species in a defined genus of mathematical objects.

Literature Arguments Supporting Alleged Self Contradiction in the Village Barber Paradox Proposition

Fraenkel and Bar-Hillel (1958) and Quine (1962) dismiss the village barber as an unrealistic person because the paradox proposition, as those authors read it, poses an impossible task for the barber. But the same authors accept the reality of Russell's class W despite the fact that definition of the class also leads to claimed self contradiction. Quine points out that the two paradoxes are exact parallels and he explains why class W is accepted but the barber is rejected. "The reason is that there has been in our habits of thought an overwhelming presumption of there being such a class but no presumption of there being such a barber." Quine's reasoning on the subject is frank but astonishing. Quine consigns the barber to limbo with ad hominem argument; then, by logical prestidigitation, he rescues class W from the barber's fate by the same kind of argument. It will be instructive to examine in detail Quine's argument sustaining the absurd conclusion that the barber shaves himself if and only if he does not shave himself.

Unfortunately, Quine offers substantially no sustaining argument. The only clue to such argument is the sentence following his statement of the paradox proposition: "Any man in this village is shaved by the barber if and only if he is not shaved by himself." That sentence leads to self contradiction because the barber is a village resident. But Quine's statement of the paradox lacks explicit assertion that every man residing in the village is shaved, either self shaved or barber shaved. By what reasoning, then, does Quine justify his use of "any man" in the quoted sentence?

Awkward Position of Permanently Bearded Men in the Barber Paradox Scenario

Quine appears to interpret the clause "those men residing in that village who do not shave themselves" as including permanently bearded men. Although such inclusion poses an impossible task for the barber, the inclusion may be a legitimate interpretation of the clause by the special language of symbolic logic (Boolean algebra). On the other hand, that interpretation leads to an abnormal dichotomous classification of village resident men regarding face hair condition. Permanently bearded men, as a separate category, are concealed in the abnormal classification; they have lost their identity as they appear in the normal classification. The normal classification first divides village resident men into (1) those who are shaved and (2) those who are permanently bearded. The shaved group is then subdivided into self shaved and notself shaved subgroups. The abnormal classification first divides the village resident men into (3) men who do shave themselves and (4) men who do not shave themselves. The (4) group may or may not be subdivided into (4a) those who are barber shaved and (4b) those who are permanently bearded, depending upon the philosophical view of the classifier.

Defenders of the alleged barber paradox prefer to interpret the clause "men who do not shave themselves" in the abnormal classification (group 4) as being restricted to village resident men who are notself shaved i.e., barber shaved. And when confronted with the unarguable fact that a permanently bearded man is one who does not shave himself, the paradox defender takes refuge in declaring that the (4b) group is superfluous, i.e., equivalent to the (4a) group (barber-shaved men). Quine's use of "any man" in his development of the barber contradiction can thus be justified, but that justification rests upon an abnormal dichotomous classification of the village resident men.

There are three distinct groups of village resident men in the barber paradox scenario: (1) self shaved men, (2) barber shaved men and (3) permanently bearded men. The three groups exhaust the family of village resident men. The problem is to provide a rational dichotomous classification of the groups into two general, one of which contains two of the groups. Paradox defenders prefer the (2) + (3) genus, but in addition they reduce that genus to the (2) group on the ground that group (3) is superfluous. The result is a two-genus family in which every member is shaved, either self shaved or barber shaved. The stage is thus rigged to ensure that the barber is shaved, thereby guaranteeing the necessary contradiction.

Author of this paper offers, instead, the (1) + (2) genus of village resident men. The members of the former genus possess the common property of being shaved; the members of the latter genus lack that common property.

First Theory of Classes with Definitions and Axioms (Zermelo, 1908)

Zermelo accepted Russell's paradox as a roadblock to development of a consistent theory of classes. He claimed that Russell's class W can't be an element of the domain of variable x in Zermelo's propositional scheme P(x) of class formation, but Zermelo was misled into concluding that class W is abnormal. It was mathematical gospel, as it is now, that membership of one class in another class is restricted to normal classes. Zermelo designed his Axiom 3 (Axiom of Separation) to circumvent class W. quoting from Zermelo's 1908 paper: "By giving us a large measure of freedom in defining new sets, Axiom 3 in a sense furnishes a substitute for the general definition that was cited in the introduction and rejected as untenable. In the first place, sets may never be independently defined by means of this axiom but must always be separated as subsets from sets already given . . . . ."

Zermelo's view of Russell's class W was modified by later workers in class theory. Class W was then accommodated, by new axioms, as a proper class. Quoting from the Introduction in a recent textbook (Takeuti and Zaring, 1982): "In G_del-Bernays set theory the classical paradoxes are avoided by recognizing two types of classes, sets and proper classes. Sets are classes that are permitted to be members of other classes. Proper classes have sets as elements but are not themselves permitted to be elements of other classes."

In a paper concerned chiefly with consistency of the continuum hypothesis, Kurt Gödel presented a 17-axiom system of class theory. He accepted the distinction between set classes and proper classes (nonsets), proved that the concept of self membered classes is untenable by means of his Axiom D, and concluded that a proper class, e.g., his universal class can never occur as an element. Gödel avoided referring to Russell's paradox. Lack of that reference seems odd to this author because Gödel implies that his universal class is the same as Russell's class W. Gödel also implies the existence of more than one proper class by referring to his universal class as an example of a proper class. Takeuti and Zaring (1982) also refer to a multiplicity of proper classes. Such multiplicity seems anomalous to this author because the individual proper classes appear to be elements of a collection.

Gleason's Development of Russell's Paradox from the Propositional Scheme (P(x) of Class Formation

Quoting from Section 7, Chapter 2 of Gleason's textbook on naive set theory (Gleason, 1966): "In Section 2-6 we established a notation which enables us to replace complicated logical statements by sets. Unfortunately, if one does this too freely, a formal contradiction can arise. There are a number of these so-called paradoxes of set theory. We shall give here what is known as Russell's paradox. . . ."

Gleason's assertion that a too-free use of the P(x) method of class formation leads to contradiction is misleading. The truth is the method always leads to contradiction. This occurs for two reasons. One, current class theory rejects the notion of self-membered classes, thus limiting set classes to notself-membered classes, the same kind as Russell's class W. Two, the P(x) method allows the domain of variable x to be co-extensive with the universal class of class names under consideration. The method runs the name of a set being formed (e.g., "G") through its procedure and obtains "for all x, x is an element of set G if and only if x is not an element of x." Replacing x with G yields the self contradiction "G is an element of G if and only if G is not an element of G."

Failure of Zermelo's P(x) method of class formation to perform as intended is remedied by adopting this author's class operator rule. That rule deletes the name of the class being formed by P(x), and only that name, from the universal class of names under consideration. The deletion occurs because the name of any object, concrete or abstract, is endowed with the class operator function by the fundamental convention of symbolic language. See Tarski, (1944 and Linsky, 1952). The convention requires use of an object's name, never the object per se, as the subject of a sentence or sentence equivalent asserting or denying anything about the object. So, if something is to be asserted about an object, e.g., that the object is an element of a set class, the object's name must be used in the asserting sentence as the subject of that sentence.

Frege's Axiom of Abstraction

Takeuti and Zaring (1982), in the introduction of their textbook on axiomatic set theory, refer to an alleged naive idea accepted by Frege (1893) in his Grundgesetze der Arithmetik. From Chapter 4 of the textbook: "We pointed out in the Introduction that one objective of axiomatic set theory is to avoid the classical paradoxes. One such paradox, the Russell paradox, arose from the naive acceptance of the idea that given any property there exists a set whose elements are the objects having that property, i.e., given a well-formed formula ∅ containing one free variable, there exists a set that contains all objects for which ∅ holds and contains no object for which ∅ fails to hold. More formally there exists a set A such that ** for all x, x is an element of set A if and only if x is not an element of x**, then in particular ** set A is an element of set A if and only if set A is not an element of set A**. This principle, called the Axiom of Abstraction, was accepted by Frege (193). In a letter to Frege (1902) Bertrand Russell pointed out that the principle leads to paradox. The idea of the collection of all objects having a specified property is so basic that we could hardly abandon it. But if it is to be retained how shall the paradox be resolved?"

The expressions enclosed by double asterisks in the above paragraph are translations of well-formed formulas written in the conventional symbols of class theory. Attention is called to the following parts of those translations: "x is not an element of x'; "set A is not an element of set A." Those parts are *ambiguous*. Variable x (an undesignated set) may be either (1) notself-membered (ordinary) or it may be (2) a member of no set, including itself. Likewise, set A may be either (1) notself-membered (ordinary) or it may be (2) a member of no set, including itself.

The author of this paper takes the stand that ambiguous language has no proper place in class theory, especially in argument used by Russell and other mathematicians to support the views that Russell's class W is self-contradictory and that Frege's Axiom of Abstraction is responsible for the so-called Russell paradox. Defenders of Russell's paradox may deny ambiguity in Russell's definition of class W and attempt to validate the alleged ambiguity with abnormal dichotomous classification of objects. But their effort is incompatible with Frege's Axiom of Abstraction.

From Frege to Gödel, van Heijenoort (1967)

The June 1902 exchange of letters between Bertrand Russell and Gotlob Frege, in German, remained unpublished until approved English translations were published in 1967 by Jean van Heijenoort. Careful reading of those translations fails, in this author's opinion, to support the literature contention that Frege's Axiom of Abstraction is responsible for Russell's paradox. Relevant excerpts from the translations are presented below.

> Russell's 16 June 1902 Letter to Frege -- There is just one point where I have encountered a difficulty. You state . . . that a function, too, can act as the indetermite element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definite collection (Menge) does not form a totality.
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> Frege's 22 June 1902 letter to Russell -- Your discovery of the contradiction caused me the greatest surprise and, I should say, consternation, since it has shaken the basis on which I intended to build arithmetic, it seems, then that transforming the generalization of an equality into an equality of courses-of-value . . . is not always permitted, that my Rule 5 is false, and that my explanations . . . are not sufficient to ensure that my combinations of signs have a meaning in all cases. I must reflect further on the matter . . . . Incidentally, it seems to me that the expression "A predicate is predicated of itself" is not exact. A predicate is as a rule a first level function, and this function requires an object as argument and cannot have itself as argument (subject). Therefore, I prefer to say, "A concept is predicated of its own extension."

It is to be noted that Frege agreed with Russell's acceptance of self predicated predicates. Frege knew that the fundamental convention of symbolic language requires all predicates in logical propositions to be notself predicated, but Russell didn't. Also, Frege's efforts to explain that point to Russell was either ignored or overlooked by Russell. The important role that Russell's paradox has played in class theory is testimony to the powerful influence of Russell's views.

Fraenkel and Bar-Hillel View of Russell's Paradox
Quoting from the authors' 1958 paper: "Let it be very clearly stated at the outset that there was absolutely nothing in the traditional treatments of logic and mathematics that could serve as a basis for the elimination of this antinomy. We think that all attempts to handle the situation without departure from traditional, i.e., pre-20th century, ways of thinking have failed thus far and are misguided in their aim." . . . . . The quoted view of Russell's paradox by Fraenkel and Bar-Hillel is a ringing endorsement of the use of ambiguous language in the pinnacle branch of mathematics.

SUMMARY

The classical Russell, Grelling and village barber paradox propositions are ambiguous. When plausibly corrected for that defect the Russell proposition remains self-contradictory. The corrected propositions lead to these logical conclusions: (1) Russell's class W can't be contained as a member in any class, including itself; (2) coined adjective "heterological" can't be grelling-classified as either autological or heterological; (3) the village barber won't be shaved, thus remaining permanently bearded. The correctly defined Grelling and barber classes, but not such Russell class, are analogs of a postulated general class S that contains member O, a class operator, which performs definable operation P upon all and only those members of class S which are notself P-operated-upon. Shifting attention from classes to class names as class members yields a new Russell class which is an analog of class S. The village barber, coined adjective "heterological" and "W", the name of class W. are class operators, each analogous to member O in class S. Logical inability of each class operator to receive its own operation is the foundation of the class operator rule. That rule supplements Zermelo's propositional scheme P(x) of class formation in three ways: (1) it eliminates Zermelo's hereditary classes; (2) it removes the distinction between set classes (sets) and proper classes (nonsets); (3) it restores Frege's Axiom of Abstraction to independent status. The P(x) scheme of class formation, currently leading to self-contradiction in every instance of its use, becomes functional as intended. Class membership is restricted to class names because no class, in terms of its members, can be a class member. Combining that conclusion with Russell's rule for collections invalidates the concept of self-membered classes. The word "class" is defined in terms of two kinds of class names: contained and uncontained in named classes. Both kinds of class names reside in the domain of variable x in the propositional scheme P(x) of class formation. Current criticism of Frege's principle of class formation (Axiom of Abstraction) as leading to Russell's paradox is shown to depend upon flawed argument. Adoption of the conclusion that class names only can be class members will require a minor change only in class theory symbolism.

Mathematics is often equated with logic, and class theory is now regarded as the pinnacle branch of mathematics. That branch ought to be devoid of ambiguous language and dichotomous classification of objects that violates Frege's principle of class formation. Since class theory now rejects the idea of self-membered classes, the idea of uncontained classes (those which are never, never contained as class members) is the principle remaining for acceptance.

Attention is called to the reasoning strategy used in current class theory to avoid the classical paradoxes. See Takeuti and Zaring (1982). Axiomatic distinction between set classes (sets) and proper classes (nonsets) is ad hoc and subjective. The author of this paper offers an objective chain of reasoning to avoid, i.e., resolve those paradoxes.

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